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Some Results on the Maximum Length of Circuits of Spread k in the d-Cube*

ROBERT J. DOUGLAS

Department of Statistics, The University of North Carolina, Chapel Hill, North Carolina 27514

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ABSTRACT

A circuit code of dimension d and spread k is a simple circuit which is formed from the vertices and edges of a d-dimensional cube and which has a certain "flatness" property making it useful as an error-limiting code. In this paper $K(d, k)$, the length of a longest d-circuit of spread k, will be exactly determined for certain pairs (d, k). The results will include a proof of a conjecture of R. C. Singleton [5].

I. INTRODUCTION

Let I^d be the graph of the d-dimensional cube $[0, 1]^d$. That is, the vertex set of I^d is $\{0, 1\}^d$ and two vertices are connected by an edge if and only if they differ in exactly one coordinate. A *d-dimensional circuit code C of spread $k \geqslant 1$* is a (simple) circuit in I^d satisfying the condition

$$d(x, y) \geqslant \min(d_c(x, y), k) \qquad \text{for all} \quad x, y \text{ in } C, \tag{1}$$

where $d(x, y)[d_c(x, y)]$ is the graph-theoretic distance between x and y in I^d [in C]; that is, $d(x, y)$ is the number of coordinates in which x and y differ, while $d_c(x, y)$ is the minimum number of edges in C used in going from x to y.

Such circuits are designed to introduce error detection into certain analog-to-digital conversion systems. The longer the code, the greater is the accuracy of the system; the greater the spread, the greater is the error-detection capability. Thus there is interest in determining $K(d, k)$, the length of a longest d-circuit of spread k. (Circuits in I^d are called *d-circuits* here.) In general, only upper and lower bounds for $K(d, k)$ are known.

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(See Chien, Freiman and Tang [1], Singleton [5], and Klee [4] for the best known general upper bounds, and Singleton [5] and Klee [2] for the best known general lower bounds.) However, there is a small list of pairs (d, k) for which the exact value of $K(d, k)$ is known. Our purpose here is to add to this list and in particular to prove a conjecture of Singleton [5].

List of Special Symbols

I^d	graph or 1-skeleton of the d-dimensional cube
$d(x, y)$	graph-theoretic distance, with respect to the graph of the d-cube, between vertices x and y
$d_c(x, y)$	graph-theoretic distance, with respect to the circuit C, between vertices x and y
$K(d, k)$	length of a longest circuit of spread k in the graph of the d-cube
$[x]$	the greatest integer less than or equal to the number x
$\tilde{S}$	cardinality of the set S
$\Leftrightarrow$	is equivalent to
$\Rightarrow$	implies
$\gamma \rightarrow \delta$	the existence of γ (in a given transition sequence) implies the existence of δ (in the same transition sequence), where γ and δ are blocks of numbers

II. Values of d and k Where $K(d, k)$ Is Known

Only for $k = 1$ has $K(d, k)$ been completely determined. It is given by

$$K(d, 1) = 2^d, \tag{1}$$

asserting that I^d always admits a Hamiltonian circuit. For arbitrary values of k, $K(d, k)$ is known when d is "small." Specifically, Singleton [5] has proved

$$K(d, k) = 2d \qquad \text{for} \quad d < \left[\frac{3k}{2}\right] + 2, \tag{2}$$

conjectured that

$$K\left(\left[\frac{3k}{2}\right] + 2, k\right) = 4k + 6 \qquad \text{for} \quad k \text{ even}, \tag{3}$$

and stated without proof that

$$K\left(\left[\frac{3k}{2}\right] + 2, k\right) = 4k + 4 \qquad \text{for} \quad k \text{ odd}. \tag{4}$$

The results (3) and (4) are proved here. For $k \leqslant 3$, and for odd $k \geqslant 9$, $K([3k/2] + 3, k)$ is known. In fact,

$$K(6, 2) = 26, K(7, 3) = 24, \tag{5}$$

and

$$K\left(\left[\frac{3k}{2}\right] + 3, k\right) = 4k + 8 \qquad \text{for odd} \quad k \geqslant 9.$$

Also

$$K(10, 5) = 28 \text{ or } 30, \qquad \text{and} \quad K(13, 7) = 36 \text{ or } 38.$$

The first of these was determined by a computer search (see Davies [2]), and the others are proved here.

III. Preliminary Remarks

In the definition of "d-circuit of spread k," (1) is easily seen to be equivalent to each of the following:

(2) $d(x, y) < k \Rightarrow d_c(x, y) = d(x, y)$ for all x and y in C;

(3) (i) $d_c(x, y) \geqslant k \Rightarrow d(x, y) \geqslant k$ for all x and y in C, and (ii) $d_c(x, y) \leqslant k \Rightarrow d(x, y) = d_c(x, y)$ for all x and y in C.

For d-circuits of length $\geqslant 2k$, $3i \Rightarrow 3ii$ and therefore $1 \Leftrightarrow 2 \Leftrightarrow 3i$. In addition, one should note that any d-circuit $v_0, \ldots, v_L(=v_0)$ can be specified by giving the starting vertex v_0 plus the *transition sequence* $a_1, \ldots, a_L$ where v_{i-1} and v_i (for $1 \leqslant i \leqslant L$) differ in the a_i-th coordinate. For two d-circuits with the same transition sequence there exists an isomorphism of I^d mapping one circuit onto the other.

Remark. *Let C be a d-circuit of spread k. Then for x and y in C, $d_c(x, y) = j \Rightarrow d(x, y) = j$ for $j \leqslant k + 1$.*

Proof: This is immediate as if x and x' are adjacent in I^d, then $d(x, y) = d(x', y) \pm 1$.

This gives us the following:

Remark. *If S is the transition sequence for a d-circuit of spread k and length L, then*: (i) $L \geqslant 2k \Rightarrow$ *In every block of k consecutive numbers of S all the numbers are distinct.* (ii) $L > 2k \Rightarrow$ *In every block of $k + 1$ consecutive numbers of S all the numbers are distinct.*

Singleton's theorem 3 and construction 3a will be frequently used, and thus are stated here:

(S3) In a d-circuit of spread k and length $N > 2d$, the existence of a block of $j \geqslant k + 2$ consecutive distinct transition numbers implies $d \geqslant k + 1 + [j/2]$.

(S3a) For odd $k \geqslant 3$, assume S is the transition sequence of a d-circuit of spread k and length N. Divide S into two segments of equal length. Then within each segment form successive blocks of $(k+1)/2$ transitions, leaving an incomplete block at the end of each segment if $k+1$ does not divide N. Alternate the new transition numbers $d+1,\ldots, d+(k+1)/2$ with the old ones in each complete block, using the new ones in the same order in each case. This construction yields a new transition sequence corresponding to a $[d+(k+1)/2]$-circuit of spread k and length $N+(k+1)[N/(k+1)]$.

IV. The Proofs of (3), (4), and (5)

Theorem 3. *$K((3k/2)+2, k) = 4k+6$ for k even.*

Proof: Let $N = 4k+6$ and $d = (3k/2)+2$. To see that $K(d,k) \geqslant N$ apply Singleton's [5, p. 599] construction 3b to the $(k+1)$-circuit of spread k and length $2(k+1)$ with $1, 2,\ldots, k+1,\ 1,\ 2,\ldots, k+1$ as transition sequence. Assume there exists a d-circuit C of spread k and length $L > N$ and consider the transition sequence S for C. We will derive a contradiction. By (S3), there does not exist a block of $k+4$ distinct numbers in S. The theorem will be proved by considering two cases.

Case I. We assume there exists a block of $k+3$ distinct numbers in S. Let S equal

$$z_1, a_1, \ldots, a_{k+3}, b_1, \ldots, b_{k+1}, c_1, \ldots, c_{k+1}, z_2, z_3, \ldots,$$

where $\omega_1 = \{a_1, \ldots, a_{k+3}\}$ gives $k+3$ distinct numbers. Write $\omega_2 = \{b_1, \ldots, b_{k+1}\}$ and $\omega_3 = \{c_1, \ldots, c_{k+1}\}$; denote by n_1 and n_2 the number of transitions appearing, respectively, once and twice in the combined ω_1, ω_2 sequence; and, finally, say transition number a_i (respectively b_i, c_i, z_i) "joins" vertex a_i' to a_i'' (respectively b_i', c_i', z_i' to b_i'', c_i'', z_i''). If $j = a+3$, then $n_1 \geqslant k$, $n_1 + n_2 \leqslant d$ and $n_1 + 2n_2 = j+k+1$, whence $2d \geqslant 2k+1+j$. Furthermore

$$(n_1 = k \quad \text{and} \quad d = n_1 + n_2) \Leftrightarrow 2d = 2k+1+j \Leftrightarrow j$$

is odd and $d = k+1+[j/2]$. But the right side holds for $j = k+3$, and consequently

$$n_1 = k, d = n_1 + n_2 \qquad \text{and} \qquad n_2 = (k/2)+2.$$

Let G be the set of all numbers occurring twice in the $\omega_1 \cup \omega_2$ block, and let

$H_i = \omega_i \sim G$ (for $i = 1, 2$). Then the cardinality of G is $(k/2) + 2$ and the cardinality of $H_1 \cup H_2$ is k. Now $d(a_1', b_{k+1}'') = k \Rightarrow c_1 \in G \Rightarrow d(b_1', c_1'') = k \Rightarrow c_2 \in H_1 \Rightarrow d(a_1', c_2'') = k \Rightarrow c_3 \in G \Rightarrow d(b_1', c_3'') = k \Rightarrow c_4 \in H_1$. Continuing in this way we conclude that

$$\{c_1, c_3, c_5, \ldots, c_{k+1}\} \subseteq G \quad \text{and} \quad \{c_2, c_4, \ldots, c_k\} \subseteq H_1.$$

Then letting $D = G \cap \omega_3$ and $H_3 = \omega_3 \sim D$ we see that the cardinality of D is $(k/2) + 1$ and the cardinality of H_3 is $k/2$.

Figure 1 summarizes the above information plus some other obvious facts:

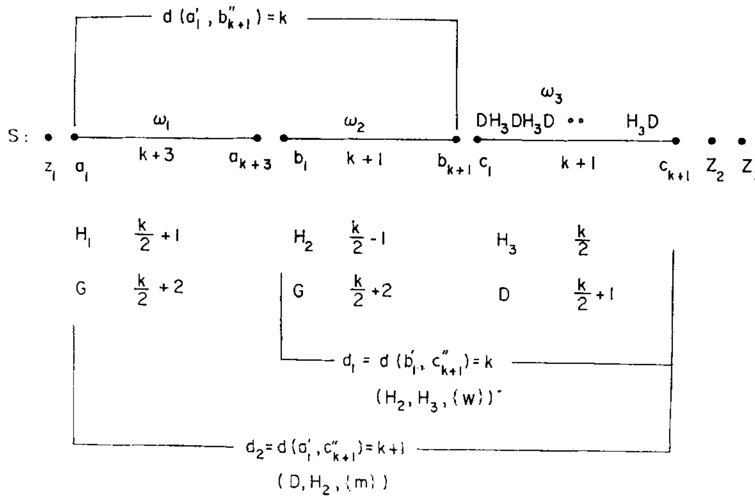
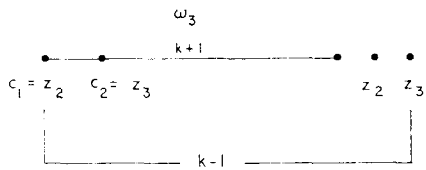


FIGURE 1

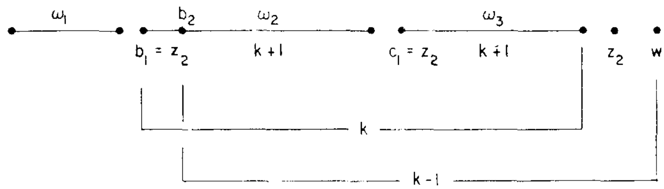
$$H_1 \cap H_2 = \phi, \quad D \subseteq G, \quad H_3 \subseteq H_1, \quad G \cap H_i = \phi \quad (i = 1, 2, 3).$$

Let $G \sim D = \{w\}$ and $H_1 \sim H_3 = \{m\}$. The transitions that "contribute" (that is, occur once) to the d_1 distance are those in H_2, H_3, and $\{w\}$. The numbers that "contribute" to the d_2 distance are those in D, H_2 and $\{m\}$.

We will now show that $z_2 = m$: Either $z_2 = m$ or (z_2 is in D and thus $z_2 = c_1$). (If not, then $d(b_1', z_2'') = k - 1$, which is a contradiction.) Hence $d(a_1', z_2'') = k$. Then $d(a_1', z_3'') = k + 1$, (otherwise $d(a_1', z_3'') = k - 1$, so $L = N$, which is a contradiction) which proves that $z_3 \in H_3$ or $z_3 = w$. Having z_3 in H_3 and z_2 in D is impossible as then $c_1 = z_2$ and $c_2 = z_3$ and thus $d(c_1', z_3'') = k - 1$, which is a contradiction.



So $z_3 = w$ or $z_2 = m$. Now $z_2 \in D \Rightarrow z_2 = c_1 \Rightarrow b_1 = z_2$. Thus $z_2 \in D \Rightarrow z_3 = w \Rightarrow d(b_2', z_3'') = k - 1$ as $w \in \omega_2$.



Hence $L < N$, which is a contradiction, and thus $z_2 \notin D$, that is, $z_2 = m$.

There does not exist a block of $k + 4$ distinct transitions, and so z_1 is in G or H_1. Now $z_1 \in H_1 \Rightarrow d(z_1', b_{k+1}'') = k - 1$, a contradiction. So $z_1 \in G$. Notice that $d(a_1', z_2'') = k$. Then $a_1 \in D \Rightarrow d(a_2', z_2'') = k - 1$, a contradiction. Either $a_1 \in G$ or else $d(a_2', b_{k+1}'') = k - 1$, a contradiction. Thus $a_1 \in G \sim D$, that is, $a_1 = w$. The fact that z_1 is adjacent to a_1 implies $z_1 \neq w$ and so $z_1 \in D$. Thus $d(a_1', z_2'') = k \Rightarrow d(z_1', z_2'') = k - 1$ and thus $L = (k + 3) + 2(k + 1) + 2 + k + 1 = 4k + 6 = N$, a contradiction.

This completes Case I.

CASE II. We assume there does not exist a block of $k + 3$ distinct transition numbers in S.

There are $d = \frac{3}{2}k + 2$ distinct transition numbers in S and these will be denoted by $1, 2, 3, \ldots, k + 1, x_1, \ldots, x_{(k/2)+1}$. We write $S = a_1, a_2, \ldots, a_L$ where transition number a_i "joins" vertex a_i' to vertex a_i''. There exists a $k + 2$ block of distinct numbers in S(or else $S = 1, 2, \ldots, k + 1, 1, 2, \ldots, k + 1$) and without loss of generality we assume $a_1, \ldots, a_{k+2}$ is such a block and is given by

$$1, x_1, 2, x_2, 3, x_3, \ldots, \frac{k}{2} + 1, x_{(k/2)+1}.$$

Denote this $k + 2$ block by "α."

LEMMA 1. $\alpha \to \alpha, x_1, (k/2) + 2$, *that is, if we have* $k + 2$ *consecutive* *distinct transition numbers in S, then the next number is the second one of the $k+2$, and the number after that is different from any of the $k+2$ numbers.*

PROOF: We have α, 1 or α, x_1 (that is, a_{k+3} equals 1 or x_1) as there is no block of $k+3$ in S. If $a_{k+3} = x_1$, then ($a_{k+4} = 1$ or else a_{k+4} is not any of the numbers $a_1 ,..., a_{k+3}$). Now α, x_1, $1 \rightarrow \alpha$, x_1, 1, 2 or $x_2 \Rightarrow d(a_1', a_{k+5}'') = k - 1$, which contradicts the assumption that $L \geqslant N$. Hence having α, x_1 implies that without loss of generality we have α, x_1, $(k/2) + 2$. Therefore it suffices to prove $\alpha \rightarrow \alpha, x_1$. Supposing that this is false, that is, α, 1 is the case, we will derive a contradiction. We obtain α, 1, x_1, 2 or x_2 (which is a contradiction as then $d(a_1', a_{k+5}'') = k-1$, so $L < N$) or α, 1, 2. The latter holds and, since $d(a_1', a_{k+4}'') = k$, without loss of generality α, 1, 2, $(k/2) + 2$ is the case. Next we have α, 1, 2, $(k/2) + 2$, x_2, 3 or x_3 (a contradiction as then $d(a_1', a_{k+7}'') = k - 1$) or α, 1, 2, $(k/2) + 2$, 3. Again the latter results, and therefore without loss of generality α, 1, 2, $(k/2) + 2$, 3, $(k/2) + 3$ holds. Repeating our reasoning gives α, 1, 2, $(k/2) + 2$, 3, $(k/2) + 3$, x_3, 4 or x_4 (a contradiction as then $d(a_1', a_{k+9}'') = k - 1$) or α, 1, 2, $(k/2) + 2$, 3, $(k/2) + 3$, 4, which results in the block α, 1, 2, $(k/2) + 2$, 3, $(k/2) + 3$, 4, $(k/2) + 4$. Continuing this process we get $a_1 ,..., a_{2k+3}$ equal to the block "ω":

$$\alpha, 1, 2, \frac{k}{2} + 2, 3, \frac{k}{2} + 3, ..., \frac{k}{2} + 1, k + 1.$$

Then we have a_L, ω where $a_L = x_{(k/2)+1}$ or $(k/2) + 1$. If $k > 2$, then $d(a_L', a_{2k}'') = k - 1$, which is a contradiction. (This yields no contradiction if $k = 2$ as then $2k = k + 2$.) If $k = 2$, then a_L, ω must equal 2, 1, x_1, 2, x_2, 1, 2, 3. An easy argument from this point will yield a contradiction to the assumption that $L > 14$.

This proves Lemma 1.

We will now repeatedly use "$\alpha \rightarrow \beta$". It is claimed that

$$\alpha \rightarrow \underbrace{\alpha, x_1, (k/2) + 2}_{\beta} \xrightarrow[\text{(i)}]{} \underbrace{\beta, x_2, (k/2) + 3}_{\gamma} \xrightarrow[\text{(ii)}]{} \gamma, x_3, (k/2) + 4.$$

PROOF OF (i): Apply "$\alpha \rightarrow \beta$" to $a_3, a_4 ,..., a_{k+4}$ to get β, x_2, y_2 where y_2 is different from any of the numbers $a_3 ,..., a_{k+4}$. Next apply "$\alpha \rightarrow \beta$" to $a_5 ,..., a_{k+6}$ to obtain β, x_2, y_2, x_3, (y_3). Therefore $y_2 = 1 \Rightarrow d(a_1', a_{k+7}'') = k - 1 \Rightarrow L < N$, which is a contradiction. Whence $y_2 \neq 1$ and without loss of generality $y_2 = (k/2) + 3$.

PROOF OF (ii): We use the same procedure as in (i). Applying "$\alpha \rightarrow \beta$"

to $a_5, \ldots, a_{k+6}$ we have γ, x_3, y_3 where y_3 is different from any of the numbers $a_5, \ldots, a_{k+6}$. Applying "$\alpha \to \beta$" again we have $\beta, x_2, (k/2) + 3, x_3, y_3, x_4, (y_4)$. Therefore $y_3 = 1$ or $2 \Rightarrow d(a_1', a_{k+9}'') = k - 1$, which is a contradiction, and hence without loss of generality $\gamma, x_3, (k/2) + 4$ is the case.

Continue this use of "$\alpha \to \beta$" to conclude that without loss of generality $a_1, \ldots, a_{2k+4}$ is equal to

$$\alpha, x_1, \frac{k}{2} + 2, x_2, \frac{k}{2} + 3, x_3, \frac{k}{2} + 4, \ldots, x_{k/2}, k + 1, x_{(k/2)+1}, \zeta$$

where ζ does not equal any number from a_{k+1} through a_{2k+2}. Let "A" denote the block $a_1, \ldots, a_{2k+3}$. Now $\zeta \neq x_i$ for any i, and in block A we have used all d transition numbers. Thus $\zeta \neq 1 \Rightarrow d(a_2', a_{2k+4}'') = k - 1 \Rightarrow L < N$, a contradiction. Therefore $\zeta = 1$. Apply "$\alpha \to \beta$" to $a_{k+3}, \ldots, a_{2k+4}$ to obtain A, 1, $(k/2) + 2$, (z_2). Now $d(a_1', a_{2k+5}'') = k - 1$, and thus $L = 2k + 4 + k = 3k + 4 < N$, which is a contradiction.

This completes Case II and hence the theorem.

THEOREM 4. *If k is odd and $d = \frac{3}{2}(k + 1)$, then $K(d, k) = 4k + 4$ and up to an isomorphism of I^d there is exactly one d-circuit of spread k and length $4k + 4$.*

PROOF: The theorem is true for $k = 1$, so assume $k \geqslant 3$. $K(d, k) \geqslant 4k + 4$ is demonstrated by applying (S3a) to the $(k + 1)$-circuit of spread k and length $2(k + 1)$ with $1, 2, \ldots, k + 1, 1, 2, \ldots, k + 1$ as transition sequence. Let $N = 4k + 4$ and let C be a d-circuit of spread k and length $L \geqslant N$. By (S3) there does not exist a block of $k + 3$ distinct numbers in S, the transition sequence for C. (Singleton then says that $L \leqslant N$ can be proved "by verifying" that a sequence of $k + 2$ numbers in S, all different, can be augmented to form only a circuit of length N. We will prove in detail that $L \leqslant N$.) Suppose $1, 2, \ldots, k + 1, x_1, \ldots, x_{(k+1)/2}$ are the d distinct transition numbers of S, where $S = a_1, \ldots, a_L$ and where transition number a_i "joins" vertex a_i' to vertex a_i''. Also assume without loss of generality that $a_1, \ldots, a_{k+2}$ is a block of $k + 2$ distinct numbers given by

$$1, x_1, 2, x_2, 3, x_3, \ldots, \frac{k+1}{2}, x_{(k+1)/2}, \frac{k+1}{2} + 1.$$

Denote this $k + 2$ block by "α".

We will show that up to an isomorphism of I^d, S must be

$$\left(1, x_1, 2, x_2, \ldots, \frac{k+1}{2}, x_{(k+1)/2}, \frac{k+1}{2}+1, x_1, \right.$$
$$\left. \frac{k+1}{2}+2, x_2, \frac{k+1}{2}+3, \ldots, k+1, x_{(k+1)/2}\right) R$$

where "R" means repeat the previous transition numbers in order.

LEMMA 2. $\alpha \to \alpha, x_1, (k+1)/2+2$.

PROOF: Now see the proof of Lemma 1. As in this proof, it suffices to show that $\alpha \to \alpha, x_1$. Say $\alpha \to \alpha$, 1 (the other possibility). Arguing exactly as in the proof of Lemma 1 we conclude that $a_1, \ldots, a_{2k+2}$ is equal to the block "ω":

$$\alpha, 1, 2, \frac{k+1}{2}+2, 3, \frac{k+1}{2}+3, \ldots, \frac{k+1}{2}, k+1.$$

Therefore a_L, ω holds where a_L is either $x_{(k+1)/2}$ or $(k+1)/2+1$. This implies $d(a_L', a_{2k+1}'') = k-1$; hence $L = 2k+2+k-1 = 3k+1 < N$, which is a contradiction.

So the lemma is proved.

Let $\beta = \alpha, x_1, (k+1)/2+2$ and as in the proof of Theorem 3, repeatedly apply "$\alpha \to \beta$" to conclude that $a_1, \ldots, a_{2k+3}$ equals

$$A, \zeta : \alpha, x_1, \frac{k+1}{2}+2, x_2, \frac{k+1}{2}+3, x_3,$$
$$\frac{k+1}{2}+4, \ldots, x_{(k+1)/2-1}, k+1, x_{(k+1)/2}, \zeta$$

where ζ is not any of the numbers from a_k through a_{2k+1}. Let "A" denote the block $a_1, \ldots, a_{2k+2}$. Then as in the proof of Theorem 3 we see that $\zeta = 1$, and so $\alpha \to A$, 1.

Next we apply "$\alpha \to A$, 1" to $a_{k+2}, \ldots, a_{2k+3}$ to obtain

$$A, 1, x_1, b_2, x_3, b_3, \ldots, x_{(k+1)/2-1}, b_{(k+1)/2}, x_{(k+1)/2}, \frac{k+1}{2}+1,$$

where the b_i are distinct and not equal to any of the numbers $a_{k+2}, \ldots, a_{2k+3}$. (So $\{b_2, \ldots, b_{(k+1)/2}\} \subseteq \{2, \ldots, (k+1)/2\}$.) Now apply "$\alpha \to A$, 1" to $a_3, \ldots, a_{k+4}$ to obtain $b_2 = 2$. (We really only use the fact "$\alpha \to \zeta = 1$.")

Next apply "$\alpha \to A$, 1" to $a_5, \ldots, a_{k+6}$ to see that $b_3 = 3$, and keep applying "$\alpha \to A$, 1" to get $b_i = i$ for $i = 2, \ldots, (k+1)/2$. We conclude that $a_1, \ldots, a_{3k+4}$ equals

$$D : A, 1, x_1, 2, x_2, \ldots, \frac{k+1}{2}, x_{(k+1)/2}, \frac{k+1}{2} + 1.$$

(Note that the last $k+2$ block of numbers in D is just the α block.) Now "$\alpha \to A$" applied to $a_{2k+3}, \ldots, a_{3k+4}$ gives $a_1, \ldots, a_{4k+4}$ equal to D, x_1, c_2, x_2, $c_3, \ldots, x_{(k+1)/2-1}$, $c_{(k+1)/2}$, $x_{(k+1)/2}$ where $c_2, \ldots, c_{(k+1)/2}$ are distinct and not equal to any numbers in α. Apply "$\alpha \to \zeta = 1$" first to $a_{k+4}, \ldots, a_{2k+5}$ to obtain $c_2 = (k+1)/2 + 2$, and then to $a_{k+6}, \ldots, a_{2k+7}$ to obtain $c_3 = (k+1)/2 + 3$. Continue this process to conclude that $c_i = (k+1)/2 + i$ for $i = 2, \ldots, (k+1)/2$. We have thus determined block $a_1, \ldots, a_{4k+4}$. This is a circuit, and so $L = N$, *proving Theorem 4.*

THEOREM 5. *$K((3k+5)/2, k) = 4k+8$ for odd $k \geq 9$. Also $K(7, 3) = 24$, $K(10, 5) = 28$ or 30, and $K(13, 7) = 36$ or 38.*

PROOF: Let $d = (3k+5)/2$ with k odd and ≥ 5. First it will be proved that $4k+8 \leq K(d,k) \leq 4k+10$. (Notice that $4k+9$ cannot be the length of a circuit in the d-cube as such a length must be even.)

Apply (S3a) to the $(k+1)$-circuit of spread k and length $2(k+1)$ with $1, 2, \ldots, k+1, 1, 2, \ldots, k+1$ as transition sequence. Then to this circuit apply Singleton's [5, p. 598] construction (1). The result is a d-circuit of spread k and length $4k+8$. Thus $K(d,k) \geq 4k+8$.

Now let C be a d-circuit of spread k and length $L > 2d$, and denote the transition sequence of C by "S." By (S3) there does not exist a block of $k+5$ distinct transitions in S. There however must exist a block of $k+2$ distinct transition numbers as $L > 2(k+1)$.

CASE I. Assume there is no $k+3$ block in S. We will show that $L \leq 4k+6$. Suppose $L > 4k+6 = N$, let $S = a_1, \ldots, a_L$, and say transition number a_i "joins" vertex a_i' to vertex a_i''. Let $1, 2, \ldots, k+1$, $x_1, \ldots, x_{(k+3)/2}$ be the distinct numbers of S, and without loss of generality say $a_1, \ldots, a_{k+2}$ equals

$$\alpha : 1, x_1, 2, x_2, \ldots, \frac{k+1}{2}, x_{(k+1)/2}, \frac{k+1}{2} + 1.$$

We have α, 1 or α, x_1. Now by arguing identically as in the proof of Lemma 2 we see that without loss of generality we have α, x_1, $(k+1)/2 + 2$. Call this block "β." By repeatedly using "$\alpha \to \beta$," and arguing exactly as in the proof of Theorem 4 we conclude that $a_1, \ldots, a_{2k+3}$ equals

$$\alpha, x_1, \frac{k+1}{2}+2, x_2, \frac{k+1}{2}+3, \ldots, x_{(k+1)/2-1}, k+1, x_{(k+1)/2}, \zeta$$

where in the present case $\zeta = 1$ *or* $x_{(k+3)/2}$. Call the $a_1, \ldots, a_{2k+2}$ block "A," and apply "$\alpha \to A$" to a_{k+2}, $a_{k+1}, \ldots, a_1$ to obtain $a_{L-(k-1)}$, $a_{L-(k-2)}, \ldots, a_L, a_1, \ldots, a_{2k+3}$ equal to

$$x_1, y_{(k+1)/2-1}, x_2, y_{(k+1)/2-2}, x_3, \ldots, y_2, x_{(k+1)/2-1}, y_1, x_{(k+1)/2}, A, \zeta$$

where no y_i is equal to any number in α. Thus

$$Y = \{y_1, \ldots, y_{(k+1)/2-1}\} \subseteq \left\{\frac{k+1}{2}+2, \frac{k+1}{2}+3, \ldots, k+1, x_{(k+3)/2}\right\}.$$

Apply "$\alpha \to A$" to $a_{k+2}, \ldots, a_{2k+3}$ to obtain $a_{L-(k-1)}, a_{L-(k-2)}, \ldots, a_L$, $a_1, \ldots, a_{3k+3}$ equal to

$$x_1, y_{(k+1)/2-1}, x_2, \ldots, y_2, x_{(k+1)/2-1}, y_1, x_{(k+1)/2};$$

$$1, x_1, 2, x_2, \ldots, \frac{k+1}{2}, x_{(k+1)/2}, \frac{k+1}{2}+1;$$

$$x_1, \frac{k+1}{2}+2, x_2, \frac{k+1}{2}+3, \ldots, x_{(k+1)/2-1}, k+1, x_{(k+1)/2}, \zeta;$$

$$x_1, z_1, x_2, z_2, x_3, \ldots, z_{(k+1)/2-1}, x_{(k+1)/2},$$

where no z_i equals any of the numbers $a_{k+2}, \ldots, a_{2k+3}$. Then

$$Z = \{z_1, \ldots, z_{(k+1)/2-1}\} \subseteq \left\{1, 2, \ldots, \frac{k+1}{2}, x_{(k+3)/2}\right\} \sim \{\zeta\}.$$

Let "γ" denote the above block of $4k+3$ numbers, and let T [respectively T_0] equal the set of all the d numbers which occur an even ($\geqslant 0$) [respectively odd] number of times in γ. If W is any set, let $\widetilde{W}$ denote the cardinality of W. Also let

$$Z' = Z \sim \{x_{(k+3)/2}\} \qquad \text{and} \qquad Y' = Y \sim \{x_{(k+3)/2}\}.$$

$$\text{If } \zeta = 1, \text{ then we have } \begin{cases} \text{(i)} \ \ \widetilde{T \cap Z'} = \dfrac{k+1}{2} - 2 \quad \text{and} \quad x_{(k+3)/2} \in Z \\ \text{or} \\ \text{(ii)} \ \ \widetilde{T \cap Z'} = \dfrac{k+1}{2} - 1 \quad \text{and} \quad x_{(k+3)/2} \notin Z. \end{cases}$$

If $\zeta = x_{(k+3)/2}$, then (iii) $\widetilde{T \cap Z'} = \frac{k+1}{2} - 1$ and $x_{(k+3)/2} \notin Z$.

Finally we have
$$\begin{cases} \text{(iv)} \quad \widetilde{T \cap Y'} = \frac{k+1}{2} - 2 \quad \text{and} \quad x_{(k+3)/2} \in Y \\ \text{or} \\ \text{(v)} \quad \widetilde{T \cap Y'} = \frac{k+1}{2} - 1 \quad \text{and} \quad x_{(k+3)/2} \notin Y. \end{cases}$$

Then if (i and iv), (i and v), (ii and iv), (iii and iv), or (iii and v) holds we have that $\tilde{T}_0 = 3$. But $3 \leqslant k - 1$ implies $L = 4k + 3 + 3 = 4k + 6 = N$, which is a contradiction.

If (ii) and (v) hold, then $\tilde{T}_0 = 1$ and thus $L = 4k + 4 < N$, which also is a contradiction.

This proves Case I.

CASE II. Assume there exists a block ω_1 of $k + 4$ distinct numbers in S. We will show that $L \leqslant 4k + 8$. Say $L > 4k + 8$. Let $S = z_1$, $a_1, \ldots, a_{k+4}, b_1, \ldots, b_{k+1}, c_1, \ldots, c_{k+1}, z_2, z_3, \ldots,$ where $\omega_1 = \{a_1, \ldots, a_{k+4}\}$ gives $k + 4$ distinct numbers, and $\omega_2 = \{b_1, \ldots, b_{k+1}\}$, and $\omega_3 = \{c_1, \ldots, c_{k+1}\}$. Further, let n_1 and n_2 denote the number of transitions appearing once and twice, respectively, in the $\omega_1 \cup \omega_2$ block. Then $n_1 \geqslant k$, $n_1 + n_2 \leqslant d$ and $n_1 + 2n_2 = 2k + 5$. By arguing as in the proof of Case I of Theorem 3 we conclude that

$$n_1 = k \qquad \text{and} \qquad d = n_1 + n_2 \qquad \text{and} \qquad n_2 = \frac{k+5}{2}.$$

Next let G be the set of all numbers appearing twice in the $\omega_1 \cup \omega_2$ block and let $H_i = \omega_i \sim G$ (for $i = 1, 2$). Then

$$\tilde{G} = \frac{k+5}{2} \qquad \text{and} \qquad \widetilde{H_1 \cup H_2} = k.$$

By exactly the same argument as in the proof of Case I of (3) we conclude that $c_1, c_3, \ldots, c_k$ are all in G and $c_2, c_4, \ldots, c_{k+1}$ are all in H_1. Let $D = G \cap \omega_3$ and $H_3 = \omega_3 \sim D$.

Figure 2 summarizes the above information plus other obvious facts:

$$H_1 \cap H_2 = \phi, \quad D \subseteq G, \quad H_3 \subseteq H_1, \quad G \cap H_i = \phi \quad (i = 1, 2, 3).$$

Write $G \sim D = \{w, w'\}$. If $z_2 \in H_3$, then $d(c_2', z_2'') = k - 1$, a contradiction, and consequently $z_2 \notin H_3$. Thus $z_2 \notin G \sim D \Rightarrow d(a_1', z_2'') =$

$k - 1 \Rightarrow L = 4k + 6$, which is a contradiction, proving that $z_2 \in G \sim D$. If $z_3 \notin D \cup (H_1 \sim H_3)$, then $d(b_1', z_3'') = k - 1$, a contradiction. Hence

$$z_2 \in G \sim D \quad \text{and} \quad z_3 \in D \cup (H_1 \sim H_3).$$

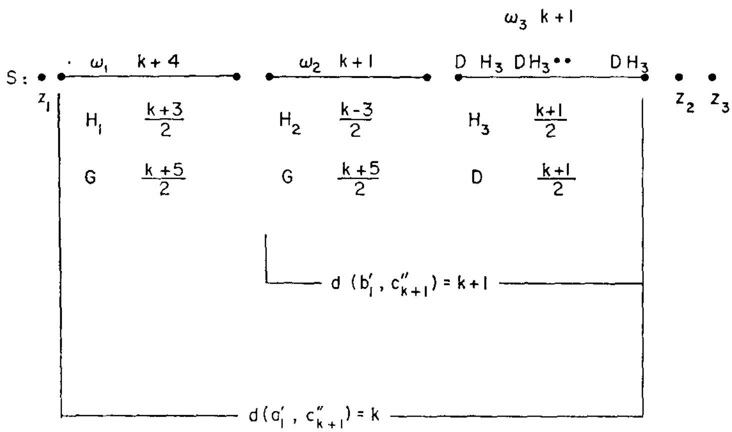


FIGURE 2

Clearly $a_1 \in G$. If $a_1 \in D$, then $d(a_2', c_{k+1}'') = k - 1$, a contradiction. Therefore $a_1 \in G \sim D$; say $a_1 = w$. Now $z_1 \in G$ as $d(a_1', b_{k+1}'') = k$, and $z_1 \notin D$ as $d(a_1', c_{k+1}'') = k$. Since $z_1 \neq w$, we must have $z_1 = w'$. Finally $z_2 = w \Rightarrow d(a_2', z_3'') = k - 1 \Rightarrow L = 4k + 6$, a contradiction, and $z_2 = w' \Rightarrow d(z_1', z_3'') = k - 1 \Rightarrow L = 4k + 8$, again a contradiction. This proves that $L \leqslant 4k + 8$ *and finishes Case II.*

CASE III. Assume there does not exist a $k + 4$ block of distinct numbers in S, but there exists a $k + 3$ block.

We will show that $L \leqslant 4k + 10$. Say $L \geqslant 4k + 10$. Let $S = e_1, \ldots, e_4, a_1, \ldots, a_{k+3}, b_1, \ldots, b_{k+1}, c_1, \ldots, c_{k+1}, d_1, \ldots, d_{k+1}, \ldots$. Let $\omega_1 = \{a_1, \ldots, a_{k+3}\}$, a block of $k + 3$ distinct numbers, $\omega_2 = \{b_1, \ldots, b_{k+1}\}$, $\omega_3 = \{c_1, \ldots, c_{k+1}\}$, $\omega_4 = \{d_1, \ldots, d_{k+1}\}$, and $\omega_5 = \{e_1, e_2, e_3, e_4\}$.

Denote by n_1 and n_2 the number of transitions appearing once and twice in the combined ω_1, ω_2 block. Therefore

$$d = \frac{3k + 5}{2}, n_1 + 2n_2 = 2k + 4, d \geqslant n_1 + n_2, n_1 \geqslant k.$$

Write $n_1 = k + p$ where $p \geqslant 0$. Then $3k + 5 = 2d \geqslant 2n_1 + 2n_2 =$

$3k + p + 4 \Rightarrow 1 \geqslant p$. Also $n_2 = (k + 4 - p)/2$. Since k is odd and $k + 4 - p$ is even, p is odd and thus $p = 1$. So

$$n_1 = k + 1, n_2 = \frac{k+3}{2} \qquad \text{and} \qquad d = n_1 + n_2 .$$

Figure 3 is given for reference and will subsequently be justified:

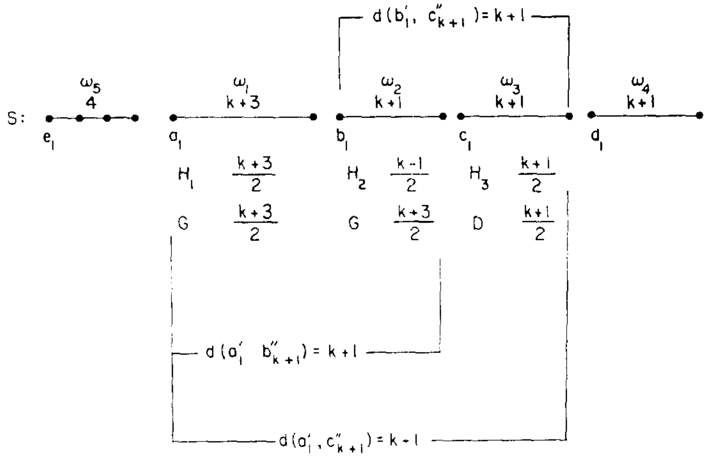


FIGURE 3

$$H_1 \cap H_2 = \phi, \quad D \subseteq G, \quad H_3 \subseteq H_1, \quad G \cap H_i = \phi \qquad \text{for all } i$$

Letting G be the set of all numbers occurring twice in the $\omega_1 \cup \omega_2$ block, and writing $H_i = \omega_i \sim G$ $(i = 1, 2)$, we have that $\tilde{G} = (k + 3)/2$, $\tilde{H}_1 = (k + 3)/2$, and $\tilde{H}_2 = (k - 1)/2$.

Now we will examine block ω_3 :

$$\widetilde{G \cap \omega_3} \geqslant \widetilde{H_1 \cap \omega_3} + \widetilde{H_2 \cap \omega_3} - 1$$

or else $d(a_1', c_{k+1}'') \leqslant k - 1$, a contradiction.

$$\widetilde{H_1 \cap \omega_3} \geqslant \widetilde{G \cap \omega_3} + \widetilde{H_2 \cap \omega_3} - 1$$

or else $d(b_1', c_{k+1}'') \leqslant k - 1$, again a contradiction. Setting $H_3 = H_1 \cap \omega_3$, $D = G \cap \omega_3$, and $x = \tilde{D}$ or $\tilde{H}_3$, we see that $x \geqslant ((k + 1) - x) - 1 \Rightarrow$

$x \geqslant (k/2) \Rightarrow x \geqslant (k+1)/2$ as k is odd. Thus $\tilde{D} \geqslant (k+1)/2$ and $\tilde{H}_3 \geqslant (k+1)/2$, and consequently (as $\tilde{\omega}_3 = k+1$) we have equality in both cases and $D \cup H_3 = \omega_3$.

Now we will examine block ω_4 : Define $H_1 \sim H_3 = \{h\}$, $G \sim D = \{w\}$, and for any set F, let $F' = F \cap \omega_4$.

$$\widetilde{(D \cup \{h\})'} \geqslant \widetilde{(H_2 \cup H_3 \cup \{w\})'} - 1$$

or else $d(b_1', d_{k+1}'') \leqslant k-1$, a contradiction.

$$\widetilde{(H_2 \cup \{w, h\})'} \geqslant \widetilde{(H_3 \cup D)'} - 1$$

or else $d(c_1', d_{k+1}'') \leqslant k-1$, again a contradiction. Therefore

$$\widetilde{(D \cup \{h\})'} \geqslant \frac{k+1}{2} \qquad \text{and} \qquad \widetilde{(H_2 \cup \{w, h\})'} \geqslant \frac{k+1}{2}.$$

Now if $h \notin \omega_4$, then both these inequalities become equalities, $\omega_4 = G \cup H_2$, $d(a_1', d_{k+1}'') = 2 \leqslant k-1$, and consequently $L = 4k+8$, a contradiction. Suppose $h \in \omega_4$ and $w \notin \omega_4$. Then $H_2 \subseteq \omega_4$ and $\tilde{D}' \geqslant (k+1)/2 - 1$. Now $\tilde{D}' = (k+1)/2 - 1 \Rightarrow \tilde{H}_3' = 1 \Rightarrow d(a_1', d_{k+1}'') = 2 \leqslant k-1 \Rightarrow L = 4k+8$, a contradiction. Also $D \subseteq \omega_4 \Rightarrow d(a_1', d_{k+1}'') = 0 \Rightarrow L = 4k+6$, another contradiction.

Hence h and w belong to ω_4.

We have

$$\tilde{D}' \geqslant \frac{k+1}{2} - 1, \tilde{H}_2' \geqslant \frac{k-1}{2} - 1, \qquad \text{and} \qquad \tilde{H}_3' \leqslant 1.$$

Hence $\tilde{H}_3' = 0 \Rightarrow d(a_1', d_{k+1}'') = 2 \Rightarrow L = 4k+8$, a contradiction. Also $\tilde{H}_3' = 1 \Rightarrow d(a_1', d_{k+1}'') = 4 \leqslant k-1 \Rightarrow L = 4k+10$.

This proves that $K(d, k) \leqslant 4k + 10$ *for odd* $k \geqslant 5$.

We now assume that $L = K(d, k) = 4k+10$, and k is odd and $\geqslant 9$. By the above, Case III holds,

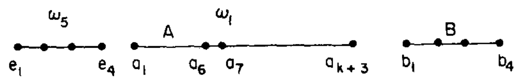
$$\{w, h\} \subseteq \omega_4, \tilde{D}' = \frac{k+1}{2} - 1, \tilde{H}_2' = \frac{k-1}{2} - 1, \qquad \text{and} \qquad \tilde{H}_3' = 1.$$

Each transition number must occur an even number of times in S. Consequently $\omega_5 = \{w, d, h_2, h_3\}$ where $h_i \in H_i$ and $d \in D$. In particular

$$\widetilde{\omega_5 \cap \omega_1} = 3.$$

Write $A = \{a_1, ..., a_6\}$ and $B = \{b_1, ..., b_4\}$. By symmetry (proceed from the "left" of the ω_1 block, instead of from the "right") we can conclude that

$$\widetilde{B \cap \omega_1} = 3.$$



Since in every block of $k + 1$ transition numbers all the numbers are distinct, we have $B \cap \omega_1 \subseteq A$, and, since $k \geqslant 9$, $A \cap \omega_5 = \phi$. Therefore $B \cap \omega_1 \cap \omega_5 = \phi$, and thus $d(e_1', b_4'') \leqslant k - 1$, a contradiction.

This proves that $K([(3k)/2] + 3, k) = 4k + 8$ *for odd* $k \geqslant 9$.

Finally let us consider $K(7, 3)$. Singleton [5] shows that $K(7, 3) \geqslant 24$. The bound $K(d, 3) \leqslant 2^d/(d - 2)$ for $d \geqslant 3$ (see Singleton [5] or Chien, Freiman, and Tang [1]) shows that $K(7, 3) < 26$, and hence $K(7, 3) \leqslant 24$ as every circuit in the d-cube has even length.

This completes the proof of Theorem 5.

V. Further Remarks

By arguments similar to Case I in the proof of Theorem 3, the following can be shown: Let S be the transition sequence of a d-circuit of spread k and length L.

(6) Assume k even, l odd $\geqslant 3$, $d = \frac{3}{2}k + (l + 1)/2$, and in S there exists a $k + l$ block of distinct transitions (equals the maximum possible). Then $k \geqslant 2l - 2 \Rightarrow L \leqslant 4k + 3l - 1$. (Also if $l = 3$, then $L = 4k + 6$.)

(7) Assume k odd $\geqslant 3$, l even, $d = \frac{3}{2}k + (l + 1)/2$, and in S there exists a $k + l$ block of distinct transitions (equals the maximum possible). Then $k \geqslant 2l + 1 \Rightarrow L \leqslant 4k + 3l + 2$. (Also if $l = 2$, then $L = 4k + 4$; if $l = 4$, then $L \leqslant 4k + 8$.)

(8) Define a simple path P in I^d to have spread $k \geqslant 1$ if

$$d(x, y) \geqslant \min(d_p(x, y), k) \qquad \text{for all} \quad x, y \text{ in } P.$$

Then, defining $P(d, k)$ to the length of a longest simple path of spread k in I^d, the following result, analogous to (2), can be shown:

$$P(d, k) = \begin{cases} 2d - k & \text{if } \; k \leqslant d \leqslant \left[\dfrac{3k}{2}\right], \\ 2d - k & \text{if } \; d = \left[\dfrac{3k}{2}\right] + 1 \quad \text{and } k \text{ is odd}, \\ 2d - k + 1 & \text{if } \; d = \left[\dfrac{3k}{2}\right] + 1 \quad \text{and } k \text{ is even}. \end{cases}$$

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